

## Experiments on isoptics by dynamic coloring

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A plane curve  $\mathcal{C}$  is given. The geometric locus of points in the plane through which passes a pair of tangents making a fixed angle  $\theta$  is called the  $\theta$ -isoptics of  $\mathcal{C}$ . We denote it by  $\text{Opt}(\mathcal{C}, \theta)$ . When  $\mathcal{C}$  is strictly convex closed curve, it defines three areas in the plane:

- Through any point inside the curve, no tangent passes.
- Through a point on the curve passes a single tangent.
- Through a point out of the curve passes a pair of tangents.

Isoptics have been studied for conics in [1], [2] (the isoptics of parabolas are arcs of hyperbolas, and the isoptics of ellipses and hyperbolas are described with spiric curves). Isoptics of open rosettes have been studied in [8]. A new approach using a DGS has been presented in [3], enabling to study isoptics of open plane curves. For general open curves, it may happen that certain areas in the plane are isopticleless.

In general, if  $C$  is a point out of the curve, the closest  $C$  is to the curve  $\mathcal{C}$ , the largest the angle between the tangents. For example, if  $\mathcal{C}$  is an ellipse, and if  $C$  is inside its director circle, then the angle is obtuse. If  $C$  is on the circle, the angle is a right angle. Otherwise, the angle is acute.

We wish to present an experimental approach to the discovery of the various areas in the plane, according to the possible angles between possible tangents. The work is based on a dynamic coloring of the plane using GeoGebra and CindyJS [7].

We begin our investigation by letting  $F(x, y) = 0$  be the equation of a convex curve. By considering an external point  $C(x_C, y_C)$  and the tangents  $t_A$  and  $t_B$  through it to the curve, we assume that there are two tangents from each point  $C$ . The tangent points are respectively  $A(x_A, y_A)$  and  $B(x_B, y_B)$ .

Clearly, the equation of a tangent at the point  $P(x, y)$  is of form

$$t_P : F'_x(x, y) \cdot (x - x_C) + F'_y(x, y) \cdot (y - y_C) = 0.$$

This can be used to express  $A$  and  $B$  with the coordinates of  $C$  without heavy computer algebra, that is, only by *derivation*, *substitutions* and *numerical equation solving* in one variable, if the following properties hold:

1.  $F$  is a polynomial of  $x$  and  $y$ .
2.  $F$  can be written in explicit form, that is, for example as  $y = f(x)$ .

For instance, when considering the example  $F(x, y) = x^2 + 2 - y$ , the formula

$$x_{A,B} = \frac{2x_C \pm \sqrt{4x_C^2 - 4y_C + 8}}{2} = x_C \pm \sqrt{x_C^2 - y_C + 2}$$

can be obtained and, from this, we immediately get  $y_A = x_A^2 + 2$  and  $y_B = y_B^2 + 2$ .

Finally, computing  $\angle ACB$  is a simple numerical operation that can be performed for each  $C$  in the plane, or in a bounding box that corresponds to the user's screen. A possible output is shown in Figure 1 where acute angles are shown in blue and obtuse angles are in red. Right angles will be obtained when the color is black, and this corresponds to the directrix of the parabola. We use a similar technique that is described in [5] and [6]. Our approach, as work-in-progress, can be generalized by embedding a computer algebra system in CindyJS—here we focus on keeping the computations as fast as possible to provide the users with immediate feedback from the computer's side.

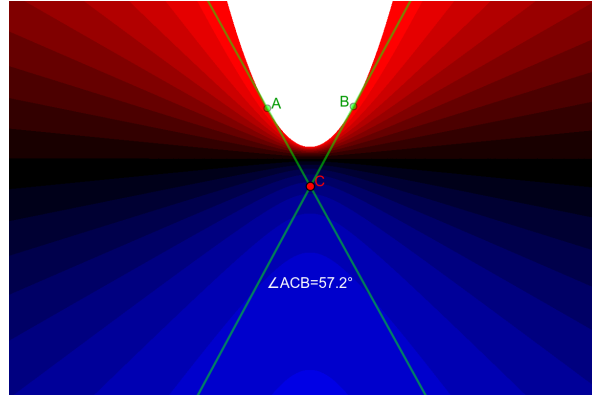


Figure 1: A CindyJS applet that presents the contour plot of isoptic angles of a parabola

Convexity of isoptics has been studied in [6]. As an example, we wish to recall that the isoptics of ellipses are ovals for obtuse angles, and non-convex closed curves for acute angles (see [1]). The same quartics (actually spirics) appear when looking for isoptics of hyperbolas. That time, the isoptic is a union of 4 disjoint arcs on both components of the spiric, as shown in Figure 2 (follow the colors).

A purely algebraic approach is possible from a theoretical point of view: if there exist points of inflexion on the isoptic  $\text{Opt}(\mathcal{C}, \theta)$ , then they are points of intersection of  $\mathcal{C}$  with its Hessian curve. A CAS may help to compute the solution of the needed system of polynomial equation,

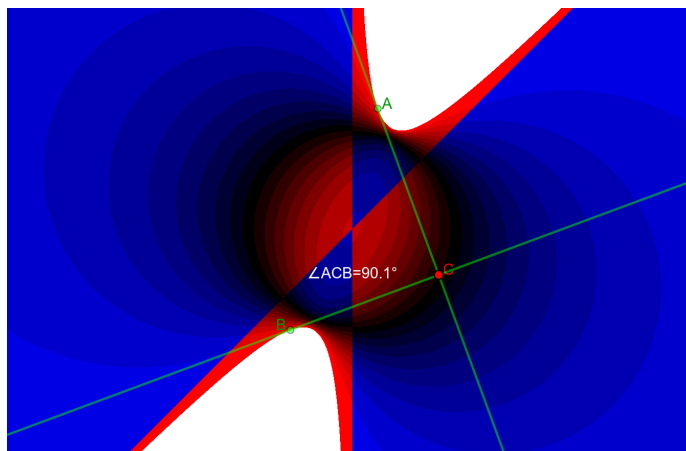


Figure 2: A contour plot of isoptic angles of the hyperbola  $F(x, y) = -x^2 + xy - 1 = 0$

but understanding and using the solution on display may be unilluminating. Working with a DGS together with a CAS may contribute to an experimental discovery of points of inflexion.

### Keywords

Isoptics, CindyJS, dynamic coloring

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