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Exploration of dual curves using dynamic geometry and computer algebra system

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This submission deals with the use of the dynamic mathematics software GeoGebra to determine the dual curve to the given curve and inspect its properties. The combination of dynamic geometry tools with computer algebra functions allows a user to take both geometric and algebraic perspectives on this issue. The dual curve to an algebraic curve is a curve born from the duality between points and lines in the projective plane. Writing the equation of a curve laying in this plane in homogeneous coordinates $[x_0, x_1, x_2]$ its tangents can be taken as points in the dual plane written in the coordinates $[y_0, y_1, y_2]$. Then the locus of these points is the dual curve to the given curve, [2].

We will show both the geometric model of the dual curve and the algebraic derivation of its equation in the talk. The geometric approach to display the dual curve shape is based on the polar reciprocity which is realized through the inversion in a circle here, [3, 4], see Fig. 1.

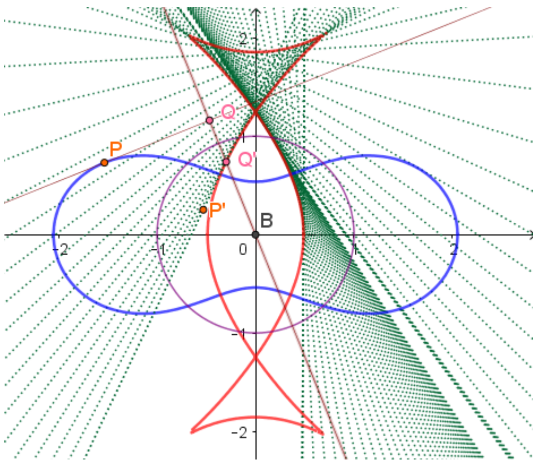

Figure 1: Dual curve to the Cassini oval as an envelope of lines dual to the points of the oval

The algebraic derivation of the dual curve equation is based on the idea that the related polynomial in indeterminates y_0, y_1, y_2 is a component of the Gröbner basis of the ideal of polynomials describing the aforesaid act of transition from a tangent line of the curve in a projective space to the point in its dual space, [5]. For example, considering the astroid with the Cartesian equation

$$27x^2y^2 + (x^2 + y^2 - 1)^3 = 0, \quad (1)$$

written in homogeneous coordinates $[x_0, x_1, x_2]$ as

$$h = x_0^6 + x_1^6 - x_2^6 + 3x_0^2x_1^4 + 3x_0^2x_2^4 + 3x_0^4x_1^2 - 3x_0^4x_2^2 + 3x_1^2x_2^4 - 3x_1^4x_2^2 + 21x_0^2x_1^2x_2^2 = 0, \quad (2)$$

the polynomial defining its dual is such a member of the Gröbner basis of the ideal of polynomials in indeterminates $x_0, x_1, x_2, y_0, y_1, y_2$

$$I = \langle y_0 - h'_{x_0}, y_1 - h'_{x_1}, y_2 - h'_{x_2}, h \rangle \tag{3}$$

that contains only indeterminates y_0, y_1, y_2. Its existence follows from the Elimination theorem, [1]. To derive the equation in GeoGebra we use the `Eliminate` command and, after transformation into the Cartesian equation

$$x^2y^2 - x^2 - y^2 = 0, \tag{4}$$

we can display the dual curve as shown in Fig. 2.

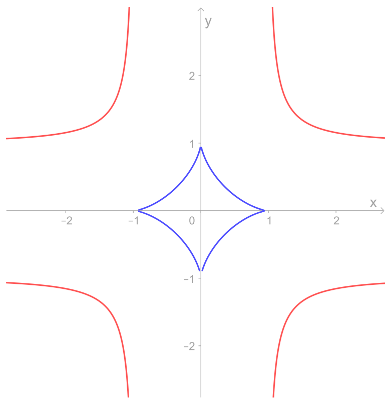

Figure 2: Dual curve (red) to the astroid (blue)

Apart from modeling the dual curve and the derivation of its equation we will also focus

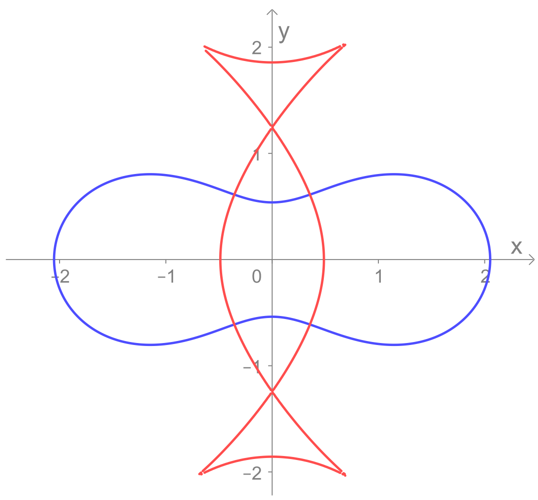

Figure 3: The Cassini oval (blue) and its dusl curve (red)

on the educational potential of this topic in the talk. The history of the notion of the dual curve is inter alia associated with the story of “the duality paradox” [3], which is worth mentioning when the concept of duality of projective space is taught. Moreover, the relation between a curve and its dual reveals concrete examples of how the duality works, [4]. See for example the correspondence between points and lines belonging to the dual curves in Figure 3, namely the correspondence between bitangents of the oval and nodes of its dual curve or the correspondence between inflexion points of the former and the cusps, more precisely tangents in them, of the latter. The utilization of dynamic geometry to explore these situations will also be presented through several particular examples.

Keywords: Computer algebra, dual curve, dynamic geometry, Gröbner basis

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