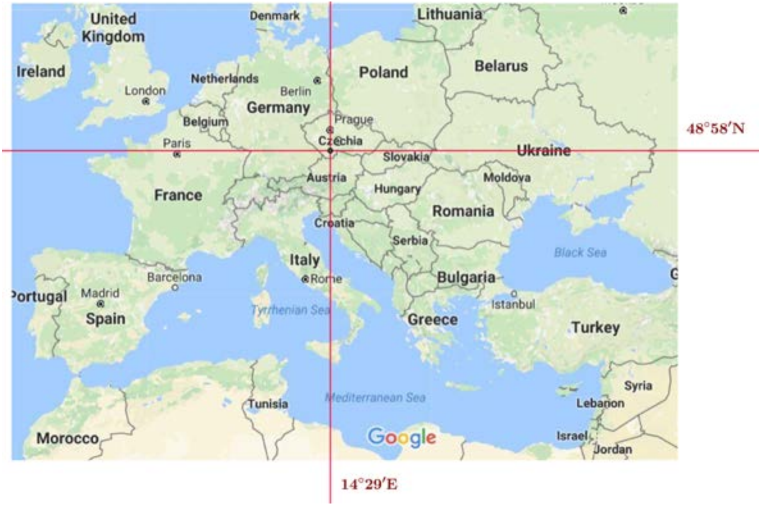
Exploration of dual curves using a dynamic geometry and computer algebra system

Roman Hašek
University of South Bohemia, Faculty of Education
České Budějovice, Czech Republic

ACA 2018, Santiago de Compostela, June 18–22, 2018

Introduction



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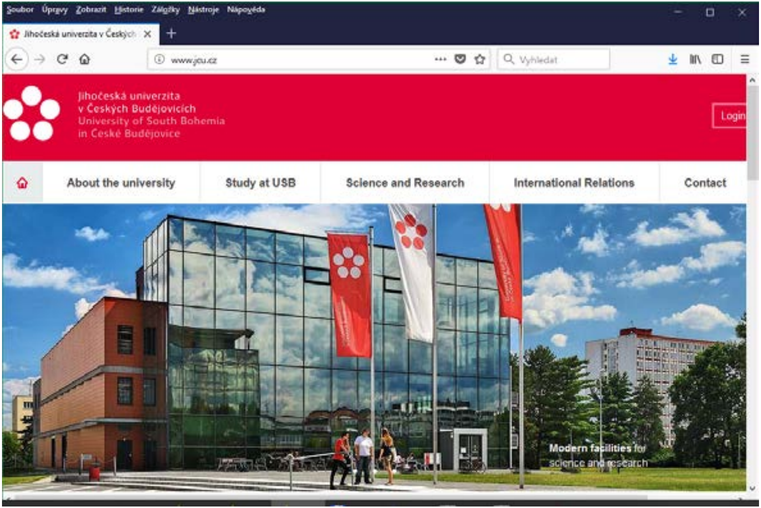
Introduction

Budweiser Budvar Beer

INTRODUCTION

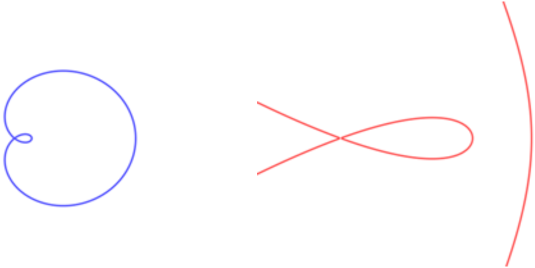


The University of South Bohemia

Mathematics teachers for elementary and secondary school

TOPIC

The use of the dynamic mathematics software GeoGebra (geogebra.org) to determine the dual curve to the given curve and inspect its properties.



Limacon and its dual curve

Principle of duality

The most characteristic feature of the projective plane (*extended Euclidean plane*).

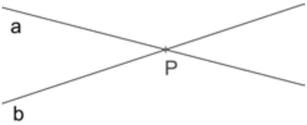
It enables us to transform points into lines and lines into points.

Dual statements

The principle of duality is reflected in dual statements. For example:

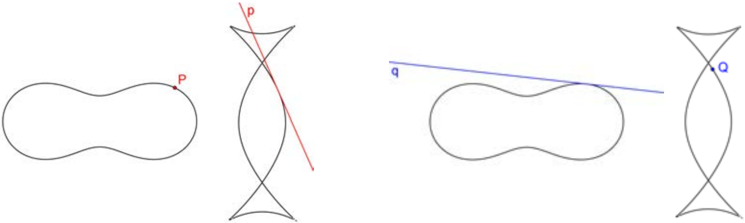
Two lines a, b pass through point P.

Two points A, B lie on a line p.



DUAL CURVES

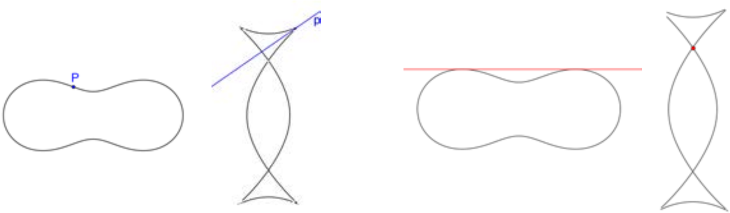
Points of one of them correspond to the *tangent lines* of the other.



Cassini oval and its dual curve

DUAL CURVES

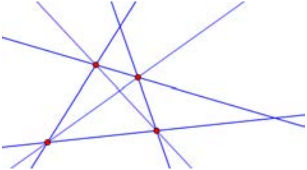
The principle of duality in practice: *inflexion points* correspond to *cusps*, *bitangents* correspond to *crunodes*.

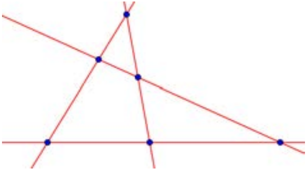


Projective geometry in maths t. t. courses

Future teachers of mathematics are introduced into the *extended Euclidean plane* $\bar{E}_2$.

- Introduction into projective plane. Principle of duality (*quadrangle* and *quadrilateral*). Dual statements.





- Homogeneous coordinates, their establishing and use. Equations of lines and conics.

$$ax + by + cz = 0, \quad ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2 = 0$$

WHY DUAL CURVES IN MATHS T. T. COURSES?

Studying the issues of existence and basic properties of dual curves using a dynamic geometry and algebraic software

- makes the topic of duality closer to the student's imagination,
- is related to other topics from the maths teacher training curriculum;
  - properties of plane curves (tangents, bitangents, singular points, inflection points, ...),
  - systems of polynomial equations and the use of CAS to solve them,
  - modelling of geometrical tasks using DGS.

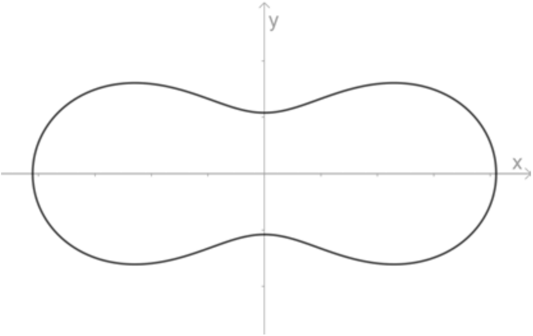
Objective

We aim to show that a unique combination of dynamic geometry and computer algebra in GeoGebra allows us to deal with the topic of dual curves, which normally is not taught in teacher training geometry courses, in a beneficial way.

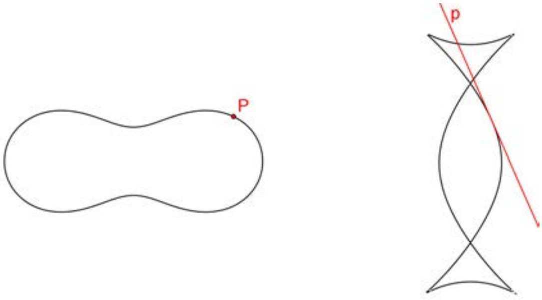
The dynamic geometry software, enriched with computer algebra functions (namely the functions for computation of Groebner bases for polynomial ideals (`GroebnerDegRevLex`) and the function `Eliminate` for eliminating variables from the system of polynomial equations), allows us to view this issue from both the geometric and algebraic point of view.

DUAL CURVE TO THE CASSINI OVAL

PROBLEM: *Determine the dual curve to the Cassini oval with the Cartesian equation* $x^4 + 2x^2y^2 + y^4 - 2a^2x^2 + 2a^2y^2 + a^4 - b^4 = 0$ for $a = 1.4, b = 1.5$.



HOW TO SOLVE IT?

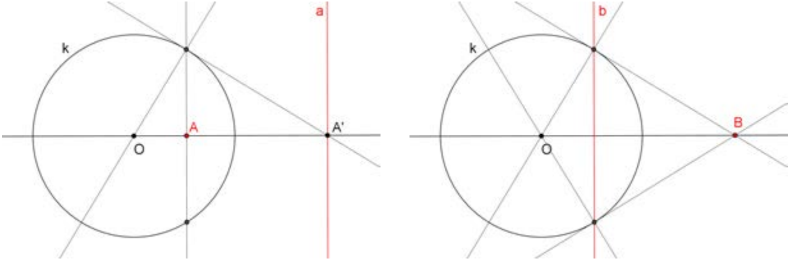


- Geometric approach – DGS
- Algebraic approach – CAS

GEOMETRIC APPROACH

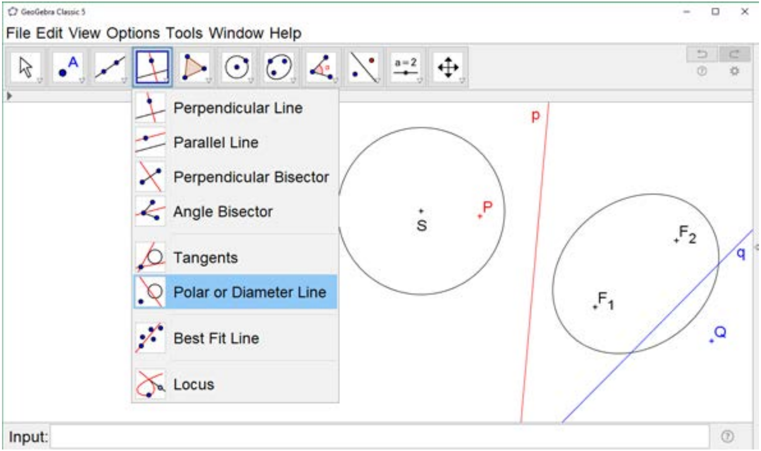
We have to define a transformation that transforms points into lines and vice versa.

$\longrightarrow$ *Reciprocation in a circle (conic section)*



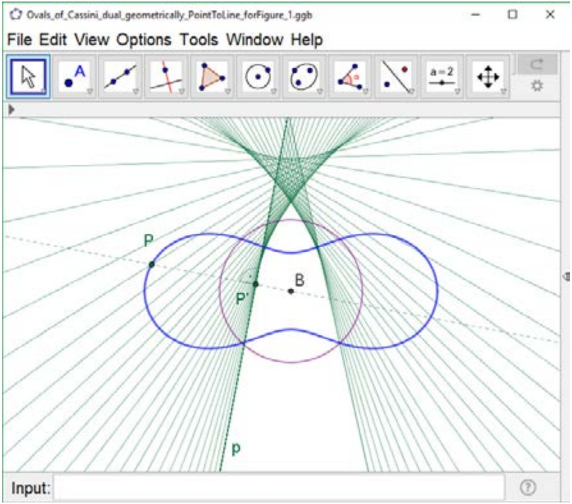
Points (*poles*) are transformed into lines (*polars*) and vice versa, lines are transformed into points.

RECIPROCATION IN GEOGEBRA



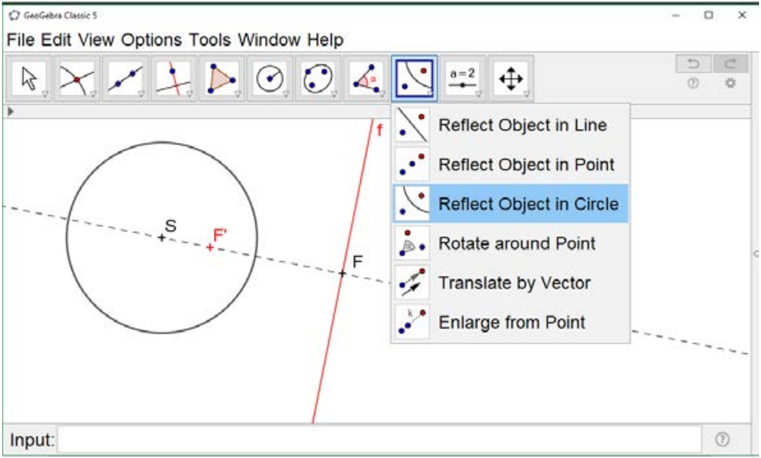
`Polar or Diameter Line` – point to line

Dual curve as an envelope of polars



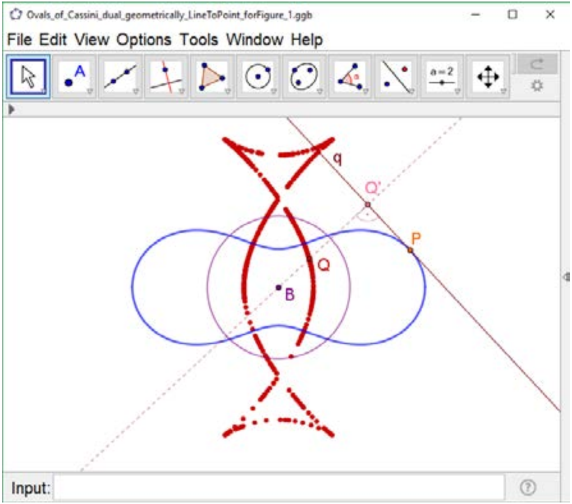
`Trace on` – tracing the polars

Inversion in GeoGebra



`Reflect Object in Circle` – line to point (image of the perpendicular foot F)

Dual curve as the locus of poles



`Trace on` – tracing the poles

ALGEBRAIC APPROACH

We move into the projective plane. Then, in homogeneous coordinates, the duality of the curves is feasible in a simple manner. If the tangent to a curve $H(x_0, x_1, x_2) = 0$ is defined by the equation

$$y_0x_0 + y_1x_1 + y_2x_2 = 0,$$

where

$$y_0 = H'_{x_0}(x_0, x_1, x_2), y_1 = H'_{x_1}(x_0, x_1, x_2), y_2 = H'_{x_2}(x_0, x_1, x_2),$$

then the point corresponding to this tangent line in the dual space is

$$\langle y_0, y_1, y_2 \rangle.$$

The dual curve is then the locus of such points.

Algebraic equation of the dual curve

The polynomial defining the dual is such a member of the Groebner basis of the ideal of polynomials in indeterminates $x_0, x_1, x_2, y_0, y_1, y_2$

$$I = \langle y_0 - h'_{x_0}, y_1 - h'_{x_1}, y_2 - h'_{x_2}, h \rangle$$

that contains only indeterminates y_0, y_1, y_2.

Its existence follows from the *Elimination theorem*.

The Elimination Theorem: Let $I \subset k[x_1, x_2, \ldots, x_n]$ be an ideal and let G be a Groebner basis of I with respect to lex order where $x_1 > x_2 > \ldots, x_n$. Then, for every $0 \leq l \leq n$, the set $G_l = G \cap k[x_1, x_2, \ldots, x_n]$ is a Groebner basis of the $l-$th elimination ideal I_l.

[D. A. Cox et al., 2007, p. 116]

SOLUTION IN GEOGEBRA CAS

GroebnerDegRevLex(<List of Polynomials>)

Computes the Gröbner basis of the list of the polynomials with respect to graded reverse lexicographical ordering of the variables (also known as *total degree reverse lexicographic ordering*, *degrevlex* or *grevlex* ordering).

> **Example:** `GroebnerDegRevLex({x^3 - y - 2, x^2 + y + 1})` yields $\{y^2 - x + 3y + 3, xy + x + y + 2, x^2 + y + 1\}$.

GroebnerDegRevLex(<List of Polynomials>, <List of Variables>)

Computes the Gröbner basis of the list of the polynomials with respect to graded reverse lexicographical ordering of the given variables (also known as *total degree reverse lexicographic ordering*, *degrevlex* or *grevlex* ordering).

> **Example:** `GroebnerDegRevLex({x^3 - y - 2, x^2 + y + 1}, {y, x})` yields $\{x^2 + y + 1, yx + y + x + 2, y^2 + 3y - x + 3\}$.

> **Note:** See also GroebnerLex and GroebnerLexDeg commands.

Eliminate(<List of Polynomials>, <List of Variables>)

Considers the algebraic equation system defined by the polynomials, and computes an equivalent system after eliminating all variables in the given list.

> **Example:** `Eliminate({x^2 + x, y^2 - x}, {x})` yields $\{y^4 + y^2\}$.

> **Note:** See also GroebnerLexDeg command.

Solution in GeoGebra CAS

CAS

1
```
O(x,y):=(x^2+y^2)^2-2 a^2 (x^2-y^2)+a^4-b^4
```
$\rightarrow\ O(x, y) := \left(x^2+y^2\right)^2 - \frac{98}{25}\left(x^2-y^2\right) - \frac{12209}{10000}$

2
```
h(x_0, x_1, x_2):=Simplify((Substitute(O(x, y),{x = x_0 / x_2, y = x_1 / x_2}) x_2⁴))
```
$\rightarrow\ h(x_0, x_1, x_2) := x_0^4 + x_1^4 - \frac{12209}{10000} x_2^4 + 2\, x_0^2\, x_1^2 - \frac{98}{25} x_0^2\, x_2^2 + \frac{98}{25} x_1^2\, x_2^2$

3
```
hx0(x_0):=Derivative(h,x_0)
```
$\rightarrow\ hx0(x_0) := 4\, x_0^3 + 4\, x_0\, x_1^2 - \frac{196}{25} x_0\, x_2^2$

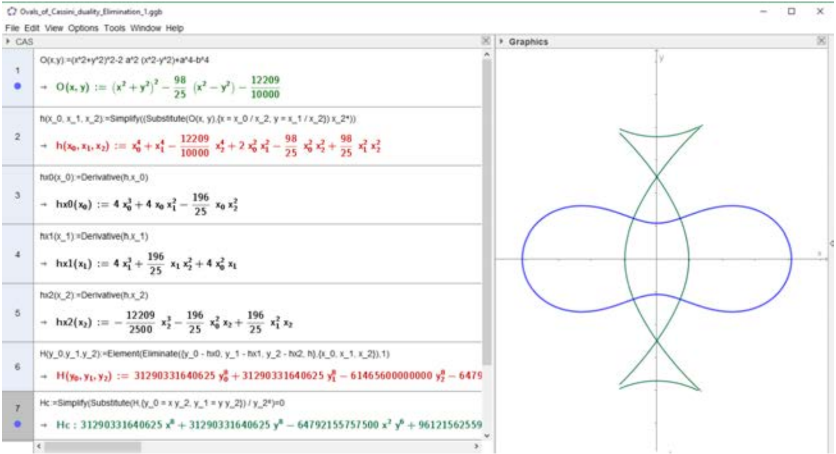
4
```
hx1(x_1):=Derivative(h,x_1)
```
$\rightarrow\ hx1(x_1) := 4\, x_1^3 + \frac{196}{25} x_1\, x_2^2 + 4\, x_0^2\, x_1$

5
```
hx2(x_2):=Derivative(h,x_2)
```
$\rightarrow\ hx2(x_2) := -\frac{12209}{2500} x_2^3 - \frac{196}{25} x_0^2\, x_2 + \frac{196}{25} x_1^2\, x_2$

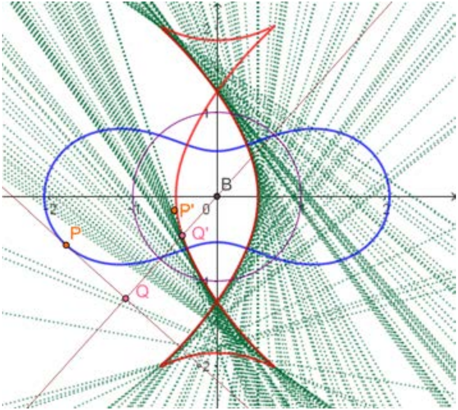
6
```
H(y_0,y_1,y_2):=Element(Eliminate({y_0 - hx0, y_1 - hx1, y_2 - hx2, h},{x_0, x_1, x_2}),1)
```
$\rightarrow\ H(y_0, y_1, y_2) := 31290331640625\, y_0^8 + 31290331640625\, y_1^8 - 61465600000000\, y_2^8 - 6479$

7
```
Hc:=Simplify(Substitute(H,{y_0 = x y_2, y_1 = y y_2}) / y_2⁸)=0
```
$\rightarrow\ Hc : 31290331640625\, x^8 + 31290331640625\, y^8 - 64792155757500\, x^2\, y^6 + 96121562559$

Solution in GeoGebra CAS

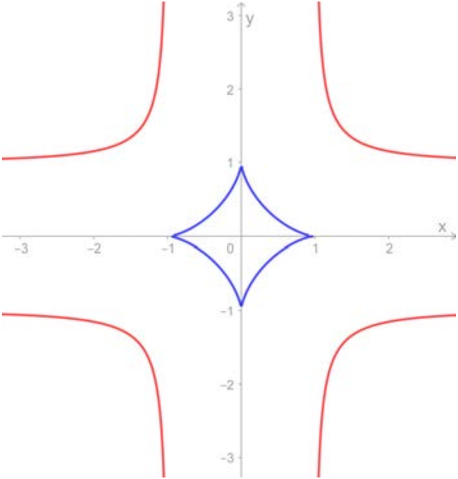


FINAL

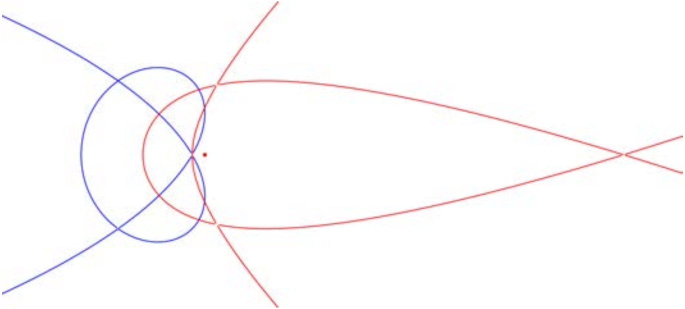


Cassini oval and its dual curve

Astroid and its dual



PRETZEL CURVE AND ITS DUAL



Conclusion

The utilization of GeoGebra makes the problem of dual curve easy to solve, related to other topics of geometry, algebra and the use of computers in mathematics education.

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hasek@pf.jcu.cz